Differential orthogonality: Laguerre and Hermite cases with applications.

J. Borrego–Morell[a,1,2], H. Pijeira–Cabrera[b,2]

[a] UNESP–Universidade Estadual Paulista, IBILCE, Brazil.
[b] Universidad Carlos III de Madrid, Spain.

Abstract

Let μ be a finite positive Borel measure supported on $\mathbb{R}$, $\mathcal{L}[f] = xf'' + (\alpha + 1 - x)f'$ with $\alpha > -1$, or $\mathcal{L}[f] = \frac{1}{2}f'' - xf'$, and m a natural number. We study algebraic, analytic and asymptotic properties of the sequence of monic polynomials $\{Q_n\}_{n>m}$ that satisfy the orthogonality relations

$$\int \mathcal{L}[Q_n](x)x^k d\mu(x) = 0 \quad \text{for all} \quad 0 \le k \le n-1.$$

We also provide a fluid dynamics model for the zeros of these polynomials.

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1. Introduction

Orthogonal polynomials with respect to a differential operator were introduced in [1] as a generalization of the notion of orthogonal polynomials. Analytic and algebraic properties of these classes of polynomials have been considered for some classes of first order differential operators in [2, 11], for a Jacobi differential operator in [4], and for differential operators of arbitrary order with polynomials coefficients in [3]. In this paper, we consider orthogonal polynomials with respect to a Laguerre or Hermite operator and a positive Borel measure μ with unbounded support on $\mathbb{R}$.

We denote by $\mathcal{L}_L$ the Laguerre and by $\mathcal{L}_H$ the Hermite differential operators on the linear space $\mathbb{P}$ of all polynomials, i.e. for all $f \in \mathbb{P}$ and $\alpha > -1$

$$\mathcal{L}_L[f] = xf'' + (1 + \alpha - x)f' = x^{-\alpha} e^x \left(x^{\alpha+1} e^{-x} f'\right)', \tag{1}$$

$$\mathcal{L}_H[f] = \frac{1}{2}f'' - xf' = \frac{1}{2}e^{x^2} \left(e^{-x^2} f'\right)'. \tag{2}$$

Each one of these second order differential operators has a system of monic polynomials which are eigenfunctions of the operator and orthogonal with respect to a measure. Let $\{L_n^\alpha\}_{n=0}^\infty$ be the monic Laguerre polynomials with $\alpha > -1$ and $\{H_n\}_{n=0}^\infty$ the monic Hermite polynomials, then

$$\langle L_n^\alpha, L_m^\alpha \rangle_L = \int L_n^\alpha(x) L_m^\alpha(x) dw_L^\alpha(x) \quad \begin{cases} = 0 & \text{if} \quad n \neq m, \\ \neq 0 & \text{if} \quad n = m, \end{cases}$$

$$\langle H_n, H_m \rangle_H = \int H_n(x) H_m(x) dw_H(x) \quad \begin{cases} = 0 & \text{if} \quad n \neq m, \\ \neq 0 & \text{if} \quad n = m, \end{cases}$$

Email addresses: jbmorell@gmail.com (J. Borrego–Morell), hpijeira@math.uc3m.es (H. Pijeira–Cabrera)
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where $dw_L^\alpha(x) = x^\alpha e^{-x} dx$ and $dw_H(x) = e^{-x^2} dx$. In addition,

$$\mathcal{L}_L[L_n^\alpha] = -nL_n^\alpha \quad \text{and} \quad \mathcal{L}_H[H_n] = -nH_n. \tag{3}$$

To unify the approach, we will denote by $\mathcal{L}$ the Laguerre or Hermite differential operator ($\mathcal{L}_L$ or $\mathcal{L}_H$) in the sequel, by dw the Laguerre or Hermite measure (dw_L^α or dw_H), by L_n the nth Laguerre or Hermite monic orthogonal polynomial (L_n^α or H_n) and by Δ the set $\mathbb{R}_+$ or $\mathbb{R}$, respectively. We will refer to one or the other depending on the case we are solving.

Let μ be a finite positive Borel measure, supported on Δ and $\{P_n\}_{n=0}^\infty$ the corresponding system of monic orthogonal polynomials, i.e.

$$\langle P_n, P_k \rangle_\mu = \int P_n(x) P_k(x) d\mu(x) \begin{cases} \neq 0 & \text{if } n = k, \\ = 0 & \text{if } n \neq k. \end{cases} \tag{4}$$

We say that Q_n is the nth monic orthogonal polynomial with respect to the pair $(\mathcal{L}, \mu)$ if Q_n has degree n and

$$\int \mathcal{L}[Q_n](x)\, x^k d\mu(x) = 0 \quad \text{for all} \quad 0 \leq k \leq n-1, \tag{5}$$

or, equivalently,

$$\mathcal{L}[Q_n] = \lambda_n P_n, \tag{6}$$

where $\lambda_n = -n$.

It was shown in [4, §2] that it is not always possible to guarantee the existence of a system of polynomials $\{Q_n\}_{n \in \mathbb{Z}_+}$ orthogonal with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$, where $\mathcal{L}^{(\alpha,\beta)}$ is the Jacobi differential operator and μ an arbitrary positive finite Borel measure. As will be shown later (cf. Propositions 1 and 2), a similar situation occurs for the case of Laguerre and Hermite operators. Let $m \in \mathbb{N}$ be fixed, a fundamental role in the existence of infinite sequences of polynomials $\{Q_n\}_{n>m}$ orthogonal with respect to the pair $(\mathcal{L}, \mu)$ is played by the class $\mathcal{P}_m(\Delta)$ defined as the family of finite positive Borel measures μ supported on Δ for which there exist a polynomial ρ of degree m, such that $\mu = (\rho)^{-1} w$.

If $\mu \in \mathcal{P}_m(\Delta)$ it is not difficult to see that if $n > m$, then

$$P_n(z) = \sum_{k=0}^{m} b_{n,n-k}\, L_{n-k}(z), \quad b_{n,n-k} = \frac{1}{\tau_{n-k}} \int P_n(x) L_{n-k}(x) dw(x), \tag{7}$$

$$\tau_n = \|L_n\|_w^2 = \int L_n^2(x) dw(x) = \begin{cases} n!\, \Gamma(n+\alpha+1) & , \mu \in \mathcal{P}_m(\mathbb{R}_+), \\ n! \sqrt{\pi} 2^{-n} & , \mu \in \mathcal{P}_m(\mathbb{R}), \end{cases} \tag{8}$$

and from (6) we obtain that the monic polynomial of degree n, for $n > m$ defined by the formula

$$\widehat{Q}_n(z) = \sum_{k=0}^{m} \frac{\lambda_n}{\lambda_{n-k}} b_{n,n-k}\, L_{n-k}(z), \tag{9}$$

is orthogonal with respect to $(\mathcal{L}, \mu)$.

Notice that from the equivalence between relations (5) and (6), the polynomial $\widehat{Q}_n + c$, $c \in \mathbb{C}$, is orthogonal with respect to $(\mathcal{L}, \mu)$ so that we do not have a unique monic orthogonal polynomial of degree n. We had a similar situation when we studied the orthogonality with respect to a Jacobi operator. A natural way to define a unique sequence would be to consider a sequence of complex numbers $\{\zeta_n\}_{n=m+1}^\infty$, and define the sequence $\{Q_n\}_{n=m+1}^\infty$ satisfying (5), as the polynomial solution of the initial value problem

$$\begin{cases} \mathcal{L}[y] = \lambda_n P_n, & n > m, \\ y(\zeta_n) = 0. \end{cases} \tag{10}$$

We say that $\{Q_n\}_{n=m+1}^\infty$ is the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$ such that $Q_n(\zeta_n) = 0$.

Notice that the initial value problem (10) has the unique polynomial solution

$$y(z) = Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n). \tag{11}$$

In this paper, we study some analytic and algebraic properties of the sequence of orthogonal polynomials with respect to a Laguerre or Hermite differential operator. In order to study the asymptotic properties of the sequence of polynomials we shall normalize them with an adequate parameter.

Let x_n be the modulus of the largest zero of the nth orthogonal polynomial with respect to μ (or w), from [12, Lemma 11 with $\lambda = 2$] for the Hermite case and [12, Coroll. (p. 191) with $\gamma = 1$] for the Laguerre case, we get

$$\lim_{n\to\infty} c_n^{-1} x_n = 1, \tag{12}$$

where c_n is usually called Mhaskar-Rakhmanov-Saff constant, here with the closed expression

$$c_n = \begin{cases} 4n & , \quad \mu \in \mathcal{P}_m(\mathbb{R}_+) \quad \text{or} \quad w(x) = x^\alpha e^{-x}, x > 0, \\ \sqrt{2n} & , \quad \mu \in \mathcal{P}_m(\mathbb{R}) \quad \text{or} \quad w(x) = e^{-x^2}, x \in \mathbb{R}. \end{cases} \tag{13}$$

Throughout this paper we denote the functions $\varphi(z) = z + \sqrt{z^2 - 1}$ and $\psi(z) = 2z - 1 + 2\sqrt{z(z-1)}$, where the branch of each root is selected from the condition $\varphi(\infty) = \infty$ and $\psi(\infty) = \infty$, respectively. Let Δ_c be the interval $[0, 1]$ in the Laguerre case and $[-1, 1]$ in the Hermite case. Let $\mathfrak{P}_n(z) = c_n^{-n} P_n(c_n z)$ be the normalized monic orthogonal polynomials with respect to a measure $\mu \in \mathcal{P}_m(\Delta)$.

To each generic polynomial q_n, let $\mu_n = n^{-1} \sum_{q_n(\omega)=0} \delta_\omega$ be the normalized root counting measure, where δ_ω is the Dirac measure with mass 1 at the point ω. From [12, Ths. 4 & 4'] we find that the limit distribution ν_w of the zero counting measure of the normalized Laguerre and Hermite polynomials is

$$d\nu_w(t) = \begin{cases} 2\pi^{-1}\sqrt{\dfrac{1-t}{t}}dt, & t \in [0, 1] \quad \text{Laguerre case,} \\ 2\pi^{-1}\sqrt{1-t^2}dt, & t \in [-1, 1] \quad \text{Hermite case.} \end{cases}$$

From [14, Chs. III & IV] we have that

$$\lim_{n\to\infty} |\mathfrak{P}_n(z)|^{\frac{1}{n}} = \begin{cases} \frac{1}{e}\ |\psi(z)|\ e^{2\Re[1/\varphi(z)]} & \mu \in \mathcal{P}_m(\mathbb{R}_+), \\ \frac{1}{2\sqrt{e}}\ |\varphi(z)|\ e^{\Re[z/\varphi(z)]} & \mu \in \mathcal{P}_m(\mathbb{R}), \end{cases} \tag{14}$$

uniformly on compact subsets $K \subset \mathbb{C} \setminus \Delta_c$.

We are interested in asymptotic properties of the normalized monic orthogonal polynomials with respect to a pair $(\mathcal{L}, \mu)$ defined by

$$\mathfrak{Q}_n(z) = \widehat{\mathfrak{Q}}_n(z) - \widehat{\mathfrak{Q}}_n(\zeta_n), \tag{15}$$

where $\widehat{\mathfrak{Q}}_n(z) = c_n^{-n}\ \widehat{Q}_n(c_n z)$. For these polynomials we prove the followings results

Theorem 1. *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m \in \mathbb{N}$. Then:*

a) If ν_n, σ_n denote the root counting measure of $\widehat{\mathfrak{Q}}_n$ and $\widehat{\mathfrak{Q}}_n'$ respectively then $\nu_n \xrightarrow{*} \nu_w$ and $\sigma_n \xrightarrow{*} \nu_w$ in the weak star sense.

b) The set of accumulation points of the zeros of $\left\{\widehat{\mathfrak{Q}}_n\right\}_{n=m+1}^{\infty}$ is Δ_c.

Theorem 2. *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\Delta)$. Then, for every compact subset K of $\mathbb{C} \setminus \Delta_c$ we have uniformly*

$$\lim_{n\to\infty} \frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}_n(z)} = \begin{cases} 1 & \mu \in \mathcal{P}_m(\mathbb{R}_+) \\ 1 & \mu \in \mathcal{P}_m(\mathbb{R}) \end{cases} \tag{16}$$

$$\lim_{n\to\infty} \left|\widehat{\mathfrak{Q}}_n(z)\right|^{\frac{1}{n}} = \begin{cases} \frac{1}{e}\ |\psi(z)|\ e^{2\Re[1/\varphi(z)]} & \mu \in \mathcal{P}_m(\mathbb{R}_+), \\ \frac{1}{2\sqrt{e}}\ |\varphi(z)|\ e^{\Re[z/\varphi(z)]} & \mu \in \mathcal{P}_m(\mathbb{R}). \end{cases} \tag{17}$$

The following result shows that the set of accumulation points of the zeros of the sequence of normalized polynomials, orthogonal with respect to $(\mathcal{L}, \mu)$ is contained in a curve.

Theorem 3. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\Delta)$. If $\{\zeta_n\}_{n=m+1}^{\infty}$ is a sequence of complex numbers with limit $\zeta \in \mathbb{C} \setminus \Delta_c$. Then:*

a) The accumulation points of zeros of the sequence $\{\mathfrak{Q}_n\}_{n=m+1}^{\infty}$ such that $\mathfrak{Q}_n(\zeta_n) = 0$ are located on the set $E = \mathcal{E}(\zeta) \bigcup \Delta_c$, where $\mathcal{E}(\zeta)$ is the curve

$$\mathcal{E}(\zeta) := \{z \in \mathbb{C} : \Psi(z) = \Psi(\zeta)\}, \tag{18}$$

$\Psi(z) = |\psi(z)|\, e^{2\Re[1/\varphi(z)]}$ for $\mu \in \mathcal{P}_m(\mathbb{R}_+)$, and $\Psi(z) = |\varphi(z)|\, e^{\Re[z/\varphi(z)]}$ for $\mu \in \mathcal{P}_m(\mathbb{R})$.

b) If $\mathfrak{d}(\zeta) = \inf_{x \in \Delta_c} |\zeta - x| > 2$ then $E = \mathcal{E}(\zeta)$ and for n sufficiently large are simple.

The relative asymptotic behavior between the sequences of polynomials $\{\mathfrak{Q}_n\}_{n>m}$ and $\{\mathfrak{P}_n\}_{n>m}$ reads as

Theorem 4. *Let $\{\zeta_n\}_{n>m}$ be a sequence of complex numbers with limit $\zeta \in \mathbb{C} \setminus \Delta_c$, $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\Delta)$ and $\{\mathfrak{Q}_n\}_{n>m}$ be the sequence of normalized monic orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$ such that $\mathfrak{Q}_n(\zeta_n) = 0$, then:*

1. *Uniformly on compact subsets of $\Omega = \{z \in \mathbb{C} : |\Psi(z)| > |\Psi(\zeta)|\}$,*
$$\frac{\mathfrak{Q}_n(z)}{\mathfrak{P}_n(z)} \underset{n\to\infty}{\rightrightarrows} 1. \tag{19}$$
2. *Uniformly on compact subsets of $\Omega = \{z \in \mathbb{C} : |\Psi(z)| < |\Psi(\zeta)|\} \setminus \Delta_c$*
$$\frac{\mathfrak{Q}_n(z)}{\mathfrak{P}_n(\zeta_n)} \underset{n\to\infty}{\rightrightarrows} -1, \tag{20}$$

where Ψ is as defined in Theorem 3. If $\mathfrak{d}(\zeta) > 2$ then (20) holds for $\Omega = \{z \in \mathbb{C} : |\Psi(z)| < |\Psi(\zeta)|\}$.

The paper continues as follows. Section 2 is dedicated to the study of existence, uniqueness and some results concerning the properties of the zeros of orthogonal polynomials with respect to the Laguerre or Hermite operators. In Sections 3 and 4 we study the asymptotic behavior of the polynomials $\widehat{\mathfrak{Q}}_n$ and $\mathfrak{Q}_n$ respectively. Finally, in Section 5 we show a fluid dynamics model for the zeros of these polynomials.

2. The polynomial Q_n

First of all, we are interested in discussing systems of polynomials such that for some $m \in \mathbb{N}$, for all $n > m$, they are solutions of (6). In order to classify those measures μ for which the existence of such sequences of orthogonal polynomials with respect to $(\mathcal{L}, \mu)$ can be guaranteed, we prove a preliminary lemma.

Lemma 5. *Let μ be a finite positive Borel measure with support contained on $\mathbb{R}$ and let $n \in \mathbb{N}$ be fixed. Then, the differential equation (6) has a monic polynomial solution Q_n of degree n, which is unique up to an additive constant, if and only if*

$$\int P_n(x)dw(x) = 0, \text{ where } P_n \text{ is as (4).} \tag{21}$$

Proof. Suppose that there exists a polynomial Q_n of degree n, such that $\mathcal{L}[Q_n] = -n\, P_n$. Then, integrating (1) or (2) with respect to the Laguerre measure on $\mathbb{R}_+$ or Hermite measure on $\mathbb{R}$ respectively we have (21).

Conversely, suppose that P_n satisfies (21). Let Q_n be the polynomial of degree n defined by $Q_n(z) = L_n(z) + \sum_{k=0}^{n-1} a_{n,k} L_k(z)$, where $a_{n,0}$ is an arbitrary constant and $a_{n,k} = \dfrac{\lambda_n}{\lambda_k \tau_k} \displaystyle\int P_n(x) L_k(x) dw(x)$, $k = 1, \ldots, n-1$. From the linearity of $\mathcal{L}[\cdot]$ and (3) we get that $\mathcal{L}[Q_n] = -n\, P_n$. ☐

From the preceding lemma, as in [4, Coroll. 2.2], we obtain

Proposition 1. *Let w be the Laguerre or Hermite measure and μ a finite positive Borel measure on Δ, such that $d\mu(x) = r(x)dw(x)$ with $r \in L^2(w)$. Then, m is the smallest natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$ if and only if r^{-1} is a polynomial of degree m.*

Proof. Suppose that m is the smallest natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$. According to Lemma 5

$$\int L_n(x)\frac{d\mu(x)}{r(x)} = \int L_n(x)dw(x) \begin{cases} = 0 & \text{if } n > m, \\ \neq 0 & \text{if } n = m. \end{cases}$$

But this is equivalent to saying that $\dfrac{1}{r(x)} = \displaystyle\sum_{k=0}^{m} c_k L_k(x)$ with $c_m \neq 0$. The converse is straightforward. □

It is possible to give another characterization, in terms of the quasi orthogonality concept, for the existence of a system of polynomials such that for all $n > m$, for some $m \in \mathbb{N}$, they are solutions of (6).

Proposition 2. *Let μ be a finite positive Borel measure on $\mathbb{R}$ and $\{P_n\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to μ. Then, m is the smallest natural number such that for each $n > m$ there exists, except for an additive constant, a unique monic polynomial Q_n, orthogonal with respect to the pair $(\mathcal{L}, \mu)$, if and only if for all $n > m$*

$$\int P_n(x)\, x^k dw(x) = 0, \quad \textit{for } k = 0, 1, \ldots, n - m,$$

i.e. the polynomial P_n is quasi-orthogonal of order $n - m + 1$ with respect to the measure w.

Proof. Assume that m is the smallest natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$. From Lemma 5 we have that (21) holds for $n > m$. From the three term recurrence relation for $\{P_n\}_{n=0}^{\infty}$

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \alpha_n^2 P_{n-1}(x),\ n \geq 1, \qquad (22)\\ & \qquad P_0(x) = 1,\ P_{-1}(x) = 0,\ \alpha_n, \beta_n \in \mathbb{R} \text{ and } \alpha_n \neq 0, \\ \text{thus} \int P_n(x)x^k dw(x) &= 0 \quad \text{for all } \ 0 \leq k < n - m, \qquad (23) \end{aligned}$$

which implies that the polynomial P_n is quasi-orthogonal of order $n - m + 1$ with respect to the measure w (Laguerre or Hermite).

Conversely, assume that m is the smallest natural number such that for $n > m$, the polynomial P_n is quasi-orthogonal of order $n - m + 1$ with respect to the measure dw. Then we have that

$$P_n(x) = L_n(x) + \sum_{k=1}^{m} d_{n-k} L_{n-k}(x),$$

which implies that for all integers $n > m$ the polynomials P_n satisfy the condition (21). From Lemma 5 we have that there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$, for all $n > m$. □

From the above proposition, we deduce in particular that the differential equation (6) has, except for an additive constant, a unique monic polynomial solution Q_n of degree n *for all the natural numbers* only if $P_n = L_n$ and $d\mu = dw$. Hence $Q_n = L_n$, the polynomial eigenfunctions of $\mathcal{L}$, whose properties are well known.

Let us continue by noting that the polynomials Q_n and $\widehat{Q}_n$ (see (9) and (11)) are primitives of the same polynomial Q'_n (or $\widehat{Q}'_n$) and

$$\int \widehat{Q}_n(x)\, x^k\, dw(x) = 0, \quad k = 0, 1, \ldots, n-m-1. \tag{24}$$

Applying classical arguments [17], it is not difficult to prove the following result, which will be used in the sequel.

Proposition 3. *The polynomial $\widehat{Q}_n$ defined by* (9) *for all $n > m$, has at least $(n-m)$ zeros and $(n-m-1)$ critical points of odd multiplicity on Δ.*

For $m = 2$ we denote by $\widetilde{\mathcal{P}}_2[\mathbb{R}]$ the class of measures of the form $d\mu = \dfrac{e^{-x^2}}{x^2 + x_1^2} dx, x_1 \neq 0$ in the Hermite case. The following proposition shows some results concerning the zeros of $\widehat{Q}_n$ and $\widehat{Q}'_n$ for measures $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$.

Proposition 4. *Assume that $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$, then the zeros of $\widehat{Q}_n$ and $\widehat{Q}'_n$ are real and simple. The critical points of Q_n interlace the zeros of P_n.*

Proof.
1.- *Laguerre case.* If $m = 1$ and $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ from Proposition 3 the polynomial $\widehat{Q}_n$ has at least $(n-1)$ real zeros of odd multiplicity on $\mathbb{R}_+$. But, $\widehat{Q}_n$ is a polynomial with real coefficients and degree n, consequently the zeros of $\widehat{Q}_n$ are real and simple. As $Q_n^{'} = \widehat{Q}_n^{'}$, from Rolle's theorem all the critical points of Q_n are real, simple, and $(n-2)$ of them are contained on $\mathbb{R}_+^* =]0, \infty[$.

Denote $G(x) = x^{\alpha+1} e^{-x} Q'_n(x)$, with $\alpha \in]-1, \infty[$. Notice that G is a real–valued, continuous and differentiable function on $\mathbb{R}_+^*$. Suppose that there exists $x \in \mathbb{R}_+^*$ such that $G(x) = 0$. As $G(0) = 0$ from Rolle's Theorem there exists $x' \in \mathbb{R}_+^*$ such that $G'(x') = 0$. But, $G'(x) = x^\alpha e^{-x} \mathcal{L}_L[Q_n] = \lambda_n x^\alpha e^{-x} P_n(x)$ and all the critical points of G are contained on $\mathbb{R}_+^*$. Hence all the critical points of Q_n belong to $\mathbb{R}_+^*$.

2.- *Hermite case.* Consider now $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$, that is, $m = 2$ and $d\mu(x) = \dfrac{e^{-x^2}}{x^2 + x_1^2} dx$, $x_1 \neq 0$. Using the relations (9) and [18, 5.6.1] we have that for $k > 1$

$$\begin{aligned}
\widehat{Q}_{2k}(z) &= L_k^{-1/2}(z^2) + \frac{k}{k-1} \frac{\langle P_{2k}, H_{2k-2}, \rangle_H}{\langle H_{2k-2}, H_{2k-2}, \rangle_H} L_{k-1}^{-1/2}(z^2), \\
\widehat{Q}_{2k+1}(z) &= z L_k^{1/2}(z^2) + \frac{2k+1}{2k-1} \frac{\langle P_{2k+1}, H_{2k-1}, \rangle_H}{\langle H_{2k-1}, H_{2k-1}, \rangle_H} z L_{k-1}^{1/2}(z^2).
\end{aligned} \tag{25}$$

As $L_n^{-1/2}(z^2), z L_n^{1/2}(z^2)$ are the $2n$ and $2n+1$ monic orthogonal polynomials of degree $2n$ and $2n+1$ respectively with respect to the measure $d\mu(x) = e^{-x^2} dx$, from (25) and [18, Th. 3.3.4] we have that the zeros of $\widehat{Q}_n, n > 2$ are real.

The statement that critical points of Q_n interlace the zeros of P_n follows by applying Rolle's theorem to the functions $G(x) = x^{\alpha+1} e^{-x} Q'_n(x)$ and $G(x) = e^{-x^2} Q'_n(x)$, for both the Laguerre and Hermite cases. $\square$

We conjecture that Proposition 4 is still valid for any measure in the class $\mathcal{P}_m(\Delta)$, $m > 1$, for the Laguerre case or $m > 2$, m even, for the Hermite case.

Finally, we find asymptotic bounds for the coefficients $b_{n,n-k}$ that define the polynomial $\widehat{Q}_n$.

Proposition 5. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\Delta)$. Then for n large enough, there exist constants C_ρ^L and C_ρ^H such that*

$$|b_{n,n-k}| = \frac{|\langle P_n, L_{n-k} \rangle_w|}{\|L_{n-k}\|_w^2} < \begin{cases} C_\rho^L\, n^k & \textit{Laguerre case,} \\ \\ C_\rho^H\, \sqrt{n^k} & \textit{Hermite case,} \end{cases}$$

for $k = 1, \ldots, m$.

Proof. Let $\rho(x)=\sum_{j=1}^{m}\rho_j x^j$ and $\rho_+=\max_{0\le j\le m}|\rho_j|$. From the Cauchy–Schwarz inequality we have

$$\begin{aligned}
|b_{n,n-k}| &\le \frac{\|P_n\|_\mu}{\|L_{n-k}\|_w^2}\sqrt{\langle \rho L_{n-k}, L_{n-k}\rangle_w} \le \frac{\|\rho L_{n-m}\|_\mu}{|\rho_m|\,\|L_{n-k}\|_w^2}\sqrt{\langle \rho L_{n-k}, L_{n-k}\rangle_w}\\
&\le \frac{\rho_+}{|\rho_m|\,\|L_{n-k}\|_w^2}\sqrt{\sum_{j=0}^{m}\left|\langle x^j, L_{n-m}^2\rangle_w\right|}\sqrt{\sum_{j=0}^{m}\left|\langle x^j, L_{n-k}^2\rangle_w\right|} \qquad (26)
\end{aligned}$$

We analyze separately the Laguerre and Hermite cases. Without loss of generality we can assume that $n>2m$.

• *Laguerre case* ($L_n=L_n^\alpha$, $\Delta=\mathbb{R}_+$ *and* $dw(x)=x^\alpha e^{-x}dx$). From [13, (III.4.9) and (I.2.9)] we have the connection formula

$$L_{n-k}^\alpha(z)=\sum_{\nu=k}^{k+j}\binom{j}{\nu-k}\frac{(n-k)!}{(n-\nu)!}L_{n-\nu}^{\alpha+j}(z),$$

then from (8) and the orthogonality

$$\begin{aligned}
\langle x^j, \left(L_{n-k}^\alpha\right)^2\rangle_L &= \sum_{\nu=k}^{k+j}\binom{j}{\nu-k}\frac{(n-k)!}{(n-\nu)!}\int \left(L_{n-\nu}^{\alpha+j}(x)\right)^2 x^{\alpha+j}e^{-x}dx,\\
&= \sum_{\nu=k}^{k+j}\binom{j}{\nu-k}(n-k)!\Gamma(n-\nu+j+\alpha+1),\\
&\le 2^j(n-k)!\Gamma(n-k+j+\alpha+1),\\
\text{and } \sum_{j=0}^{m}\langle x^j, L_{n-k}^2\rangle_w &\le (n-k)!\sum_{j=0}^{m}2^j\Gamma(n-k+j+\alpha+1),\\
&\le (2^{m+1}-1)(n-k)!\Gamma(n-k+m+\alpha+1).
\end{aligned}$$

Hence, from (26), (8) and n large enough

$$\begin{aligned}
|b_{n,n-k}| &\le \frac{\rho_+(2^{m+1}-1)}{|\rho_m|}\sqrt{\frac{(n-m)!\Gamma(n+\alpha+1)\Gamma(n+m-k+\alpha+1)}{(n-k)!\Gamma^2(n-k+\alpha+1)}},\\
&\le \frac{\rho_+(2^{m+1}-1)}{|\rho_m|}\sqrt{\frac{(n+\alpha)^{k+m}}{(n-m)^{m-k}}} \le \frac{\rho_+2^m(2^{m+1}-1)}{|\rho_m|}\,n^k.
\end{aligned}$$

• *Hermite case* ($L_n=H_n$, $\Delta=\mathbb{R}$ *and* $dw(x)=e^{-x^2}dx$). By the symmetry property of the Hermite polynomials, if ν is an odd number

$$\int x^\nu\, H_{n-k}^2(x)dw(x)=0.$$

Hence, from (26)

$$|b_{n,n-k}|\le \frac{\rho_+}{|\rho_m|\,\|H_{n-k}\|_w^2}\sqrt{\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\|x^j\,H_{n-m}\|_w^2}\sqrt{\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\|x^j\,H_{n-k}\|_w^2},$$

where for all $x\in\mathbb{R}$, the symbol $\lfloor x\rfloor$ denote the largest integer less than or equal to x. As it is well known (cf. [18, (5.5.6) and (5.5.8)]), the Hermite polynomials satisfy the recurrence relation $zH_n(z)=H_{n+1}(z)+\frac{n}{2}H_{n-1}(z)$, from which we get by induction on j

$$z^jH_n(z)=\sum_{\nu=0}^{j}\sigma_{j,\nu}(n)H_{n+j-2\nu}(z), \qquad (27)$$

where $\sigma_{j,\nu}(n)$ is a polynomial in n of degree equal to ν and leading coefficient $2^{-\nu}\binom{j}{v}$ (i.e. $\sigma_{j,\nu}(n) = 2^{-\nu}\binom{j}{v}n^{\nu}+\cdots$). Hence, from (8), for n large enough

$$
\begin{aligned}
\|x^j H_{n-k}\|_w^2 &= \sum_{\nu=0}^{j} \sigma_{j,\nu}^2(n-k)\|H_{n-k+j-2\nu}\|_w^2, \\
&\leq \frac{\sqrt{\pi}\,(n-k-j)!}{2^{n-k+j}}\left(\sum_{\nu=0}^{j} 2^{2\nu}\sigma_{j,\nu}^2(n-k)(n-k+j)^{2j-2\nu}\right), \\
&\leq \frac{2\sqrt{\pi}\,(n-k-j)!(n-k)^{2j}}{2^{n-k}}\binom{2j}{j},
\end{aligned}
$$

with $j = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor$, therefore

$$
\begin{aligned}
|b_{n,n-k}| &\leq \frac{\rho_+\, 2^{n-k}}{\sqrt{\pi}|\rho_m|\,(n-k)!}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\|x^j\,H_{n-m}\|_w^2}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\|x^j\,H_{n-k}\|_w^2} \\
&\leq \frac{2m!\rho_+}{|\rho_m|}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}(n-m)^{2j}\frac{(n-m-j)!}{(n-k)!}}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}(n-k)^{2j}\frac{(n-k-j)!}{(n-k)!}} \\
&\leq \frac{2m!\rho_+}{|\rho_m|}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\frac{(n-m)^{2j}}{(n-m-j)^{m+j-k}}}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\frac{(n-k)^{2j}}{(n-m-j)^{j}}} \\
&\leq \frac{2m!\rho_+}{|\rho_m|}\sqrt{8m(n-k)^{-\lfloor \frac{m}{2}\rfloor}}\sqrt{2m(n-k)^{\lfloor \frac{m}{2}\rfloor}}\,n^k = \frac{8m(m)!\rho_+}{|\rho_m|}\,n^k.
\end{aligned}
$$

□

3. The polynomial $\widehat{\mathfrak{Q}}_n$

In this section we prove asymptotic properties of the normalized monic orthogonal polynomials with respect to a Laguerre or Hermite differential operator. We recall that as in Section 1, Δ_c denotes the interval $[0,1]$ in the Laguerre case and $[-1,1]$ in the Hermite case, and the sequence of real numbers $\{c_n\}_{n=1}^{\infty}$ is given by (13). Set $\mathfrak{L}_{n,\nu}(z) = c_n^{-\nu}\,L_\nu(c_n z); \mathfrak{L}_n(z) \equiv \mathfrak{L}_{n,n}(z)$ and $\mathfrak{P}_{n,\nu}(z) = c_n^{-\nu}\,P_\nu(c_n z); \mathfrak{P}_n(z) \equiv \mathfrak{P}_{n,n}(z)$.

We prove now some preliminary lemmas.

Lemma 6. *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\Delta)$ and ζ such that $\widehat{\mathfrak{Q}}_n(\zeta) = 0$. Then for all n sufficiently large $d_c(\zeta) < 2\varpi_c$, where*

$$
\varpi_c = \begin{cases} 1 + 2^{-1}\,C_\rho^L & \textit{Laguerre case,} \\ 1 + \sqrt{2}\,C_\rho^H & \textit{Hermite case,} \end{cases}
$$

$d_c(z) = \min_{x\in\Delta_c}|z-x|$, *and C_ρ^L and C_ρ^H are the same constants of Proposition 5.*

Proof. For each fixed $n > m$, we have that

$$
x_n^{-n}\widehat{Q}_n(x_n z) = \sum_{k=0}^{m}\frac{\lambda_n\,b_{n,n-k}}{x_n^k\lambda_{n-k}}\,x_n^{-n+k}L_{n-k}(x_n z),
$$

where x_n is the zero of the largest modulus of L_n. It follows that the smallest interval containing the zeros of $\{x_n^{-k}L_k(x_nz)\}_{k=0}^n$ is Δ_c. Hence, if ζ is such that $\widehat{Q}_n(x_n\zeta)=0$, from [15, Coroll. 1], Proposition 5, (13), and (12) we have,

$$d_c(\zeta) \quad\leq\quad 1+\max_{1\leq k\leq m}\left|\frac{\lambda_n\, b_{n,n-k}}{x_n^k\lambda_{n-k}}\right| < 1+2\max_{1\leq k\leq m}\left|\frac{b_{n,n-k}}{x_n^k}\right| \leq \varpi_c, \tag{28}$$

where

$$\varpi_c=\begin{cases} 1+2^{-1}\,C_\rho^L & \text{Laguerre case,}\\ 1+\sqrt{2}\,C_\rho^H & \text{Hermite case.}\end{cases}$$

Notice that $\widehat{\mathfrak{Q}}_n\left(\frac{x_n}{c_n}z\right)=c_n^{-n}\widehat{Q}_n(x_nz)$; therefore, if ζ is such that $\widehat{Q}_n(x_n\zeta)=0$ then $\zeta^*=\frac{x_n}{c_n}\zeta$ is such that $\widehat{\mathfrak{Q}}_n(\zeta^*)=0$. From (12) and (13) we have that for n large, $\left|\frac{x_n}{c_n}\right|<2$. Using now (28) we obtain the lemma. □

If $\{\Pi_n\}_{n=0}^\infty$ is a sequence of orthogonal polynomials with respect to either the measures μ or w we denote by $\{\mathfrak{k}_n\}_{n=0}^\infty$ the sequence of monic normalized polynomials, that is,

$$\mathfrak{k}_n(z)=c_n^{-n}\Pi_n(c_nz) \;\text{ and }\; \mathfrak{k}_{n,\nu}(z)=c_n^{-\nu}\Pi_\nu(c_nz). \tag{29}$$

From the interlacing property of the zeros of consecutive orthogonal polynomials, if K is a compact subset of $\mathbb{C}\setminus\Delta_c$ it follows that there exist a constant M_* such that for n large enough

$$\left|\frac{\mathfrak{k}_{n,n-k}(z)}{\mathfrak{k}_n(z)}\right|<M_k\leq M_*,\quad k=1,\ldots,m, \tag{30}$$

uniformly on $z\in K$, where $M_k=2\sup_{\substack{z\in K\\ x\in\Delta_c}}|z-x|^{-k}$, $M_*=\max\{M_1,\ldots,M_m\}$.

The following lemma is needed to study the modulus of the sequence $\left\{\frac{\mathfrak{P}_n}{\mathfrak{L}_n}\right\}_{n=0}^\infty$.

Lemma 7. *Suppose that $m\in\mathbb{N}$ is fixed, and $K\subset\mathbb{C}\setminus\Delta_c$ a compact subset. Then, for n sufficiently large*

$$\left|\left(\frac{c_{n+m}}{c_n}\right)^n\frac{\mathfrak{k}_n(z)}{\mathfrak{k}_n(\frac{c_{n+m}}{c_n}z)}\right| \quad<\quad 3^{\frac{2m}{d}},\qquad n>n_0,\forall z\in K, \tag{31}$$

where $d=\inf_{\substack{z\in K\\ x\in\Delta_c}}|z-x|$ and $\mathfrak{k}_n$ as in (29).

Proof. Let us define the monic polynomial $\mathfrak{k}_n^*(z)=\left(\frac{c_n}{c_{n+m}}\right)^n\mathfrak{k}_n\left(\frac{c_{n+m}}{c_n}z\right)$. We have that (31) is equivalent to proving that

$$\left|\frac{\mathfrak{k}_n(z)}{\mathfrak{k}_n^*(z)}\right|\leq 3^{\frac{2m}{d}},\qquad n>n_0,\forall z\in K.$$

If $\{z_{k,n}^*\}_{k=1}^n$, $\{z_{k,n}\}_{k=1}^n$ denotes the zeros of the polynomials $\mathfrak{k}_n^*$, $\mathfrak{k}_n$ respectively, we have the relation $z_{k,n}^*=\frac{c_n}{c_{n+m}}z_{k,n}$, $k=1,\ldots,n$. If we denote $k_n=\frac{c_n}{c_{n+m}}$, we have, for all n sufficiently large

$$\begin{aligned}\left|\frac{\mathfrak{k}_n(z)}{\mathfrak{k}_n^*(z)}\right| &\leq \left|\prod_{k=1}^n\left(1+\frac{(k_n-1)z_{k,n}}{z-k_nz_{k,n}}\right)\right|\leq\prod_{k=1}^n\left(1+|k_n-1|\left|\frac{z_{k,n}}{z-k_nz_{k,n}}\right|\right)\\ &\leq \prod_{k=1}^n\left(1+\frac{2|k_n-1|}{d}\right)\leq\left(1+\frac{2|k_n-1|}{d}\right)^n<3^{\frac{2n|k_n-1|}{d}}\leq 3^{\frac{2m}{d}},\end{aligned} \tag{32}$$

where $d = \inf_{\substack{z\in K\\ x\in\Delta_c}} |z-x|$. $\square$

We prove now that the modulus of the sequence $\left\{\frac{\mathfrak{P}_n}{\mathfrak{L}_n}\right\}_{n=0}^{\infty}$ is uniformly bounded from above and below in the interior of $\mathbb{C}\setminus\Delta_c$.

Lemma 8. *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m\in\mathbb{N}$ and $K\subset\mathbb{C}\setminus\Delta_c$ a compact subset. Then, for all n sufficiently large there exists a constant C^* such that*

$$\left|\frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)}\right| \le C^*, \quad n>n_0, \forall z\in K.$$

Proof. From Relation (7) we deduce that $\frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)} = 1+\sum_{k=1}^{m}\frac{b_{n,n-k}}{c_n^k}\frac{\mathfrak{L}_{n,n-k}(z)}{\mathfrak{L}_n(z)}$. Hence, from Proposition 5, and Lemma 7 we deduce that for n large enough

$$\left|\frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)}\right| \le 1+\sum_{k=1}^{m} C_\rho\left|\frac{\mathfrak{L}_{n,n-k}(z)}{\mathfrak{L}_n(z)}\right|, \tag{33}$$

Using (33) and (30) we deduce the lemma. $\square$

Lemma 9. *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m\in\mathbb{N}$ and $K\subset\mathbb{C}\setminus\Delta_c$ is a compact subset. Then, for all n sufficiently large there exists a constant C such that*

$$C\le\left|\frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)}\right|, \quad n>n_0, \forall z\in K.$$

Proof. We have that $\rho(z)L_n(z) = \sum_{k=0}^{m}\mathfrak{b}_{n,n-k}P_{n+m-k}(z)$, where $\mathfrak{b}_{n,n-k} = \frac{\int L_n(x)P_{n+m-k}(x)\rho(x)d\mu(x)}{\|P_{n+m-k}(x)\|_\mu^2}$, or equivalently,

$$\frac{\rho(c_{n+m}z)}{c_{n+m}^m}\left(\frac{c_n}{c_{n+m}}\right)^n\frac{\mathfrak{L}_n(\frac{c_{n+m}}{c_n}z)}{\mathfrak{L}_n(z)}\frac{\mathfrak{L}_n(z)}{\mathfrak{P}_{n+m}(z)} = \sum_{k=0}^{m}\frac{\mathfrak{b}_{n,n-k}}{c_{n+m}^k}\frac{\mathfrak{P}_{n+m,n+m-k}(z)}{\mathfrak{P}_{n+m}(z)}. \tag{34}$$

From the Cauchy Schwartz inequality we have that

$$|\mathfrak{b}_{n,n-k}| \le \frac{\left(\int L_n^2(x)dw(x)\right)^{1/2}(\int P_{n+m-k}^2(x)dw(x))^{1/2}}{\|P_{n+m-k}\|_\mu^2} = \frac{\|L_n\|_w\|P_{n+m-k}\|_w}{\|P_{n+m-k}\|_\mu^2}.$$

Using an infinite–finite range inequality for the case in which w is a Laguerre weight, cf. [14], we have that there exists a constant k_L such that for all n large enough

$$\frac{k_L}{n^m}\int_0^\infty L_n^2(x)dw(x) \le \frac{k_{0,L}}{(4n)^m}\int_0^\infty P_n^2(x)dw(x) \le \frac{1}{\rho_+(4n)^m}\int_0^{4n}P_n^2(x)dw(x) \le \int_0^\infty P_n^2(x)d\mu(x),$$

where $\rho_+ = \max_{0\le j\le m}|\rho_j|$. Analogously, for the case of an Hermite weight, for all n large enough, we have that there exists a constant k_H such that

$$\frac{k_H}{n^{m/2}}\int_{-\infty}^\infty L_n^2(x)dw(x) \le \frac{k_{0,H}}{(2n)^{m/2}}\int_{-\infty}^\infty P_n^2(x)dw(x) \le \frac{1}{\rho_+(2n)^{m/2}}\int_{-\sqrt{2n}}^{\sqrt{2n}}P_n^2(x)dw(x) \le \int_{-\infty}^\infty P_n^2(x)d\mu(x),$$

Hence, for all n large enough

$$\begin{aligned} \|P_n\|_\mu^2 &\geq k_L n^{-m}\|L_n\|_w^2, \quad \text{Laguerre case,} \\ \|P_n\|_\mu^2 &\geq k_H n^{-m/2}\|L_n\|_w^2, \quad \text{Hermite case.} \end{aligned} \tag{35}$$

From (7) and Proposition 5 we deduce that for n large enough, there exists a constant k_1 such that

$$\|P_n\|_w \leq k_1\|L_n\|_w. \tag{36}$$

Inequalities (35) and (36) give us that there exists a constant M^* such that for all n large enough

$$\frac{|\mathfrak{b}_{n,n-k}|}{c_{n+m}^k} \leq M^*, \quad 1 \leq k \leq m. \tag{37}$$

From (30) it follows that there exists a constant M_* such that for all $z \in K$

$$\left|\frac{\mathfrak{P}_{n+m,n+m-k}(z)}{\mathfrak{P}_{n+m}(z)}\right| < M_*, \quad k = 1,\ldots,m. \tag{38}$$

Using Lemma 7, (34), (37) and (38) we obtain

$$\left|\frac{\rho(c_{n+m}z)}{c_{n+m}^m}\right|\left|\frac{\mathfrak{L}_n(z)}{\mathfrak{P}_{n+m}(z)}\right| \leq 3^{\frac{2m}{d}}\left(1 + m\,M^*\,M_*\right), \tag{39}$$

with d as in Lemma 7. Hence, from (30), (38), (39) and Lemma 7 we obtain that for all n sufficiently large there exists $M > 0$ such that

$$\left|\frac{\rho(c_{n+m}z)}{c_{n+m}^m}\right|\left|\frac{\mathfrak{L}_n(z)}{\mathfrak{P}_n(z)}\right| \leq M, \quad \forall z \in K. \tag{40}$$

Let us denote by $\{z_k\}_{k=1}^m$ the roots of the polynomial ρ, and $d^* = \inf_{z\in K}|z|$. Let us choose ε so that for n large enough $\left|\frac{z_k}{c_{n+m}}\right| < \varepsilon < d^*, k = 1,\ldots,m$. Hence,

$$(d^*-\varepsilon)^m \leq \prod_{k=1}^m\left(|z| - \left|\frac{z_k}{c_{n+m}}\right|\right) \leq \prod_{k=1}^m\left|\left(z - \frac{z_k}{c_{n+m}}\right)\right| = \left|\frac{\rho(c_{n+m}z)}{c_{n+m}^m}\right|. \tag{41}$$

Therefore, from (40) and (41), for all n large enough we have that

$$\left|\frac{\mathfrak{L}_n(z)}{\mathfrak{P}_n(z)}\right| \leq \frac{M}{(d^*-\varepsilon)^m}, \quad \forall z \in K,$$

and this proves the lemma. □

Lemma 10. *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m \in \mathbb{N}$ and $K \subset \mathbb{C}\setminus\Delta_c$ is a compact subset. Then,*

$$\left|\frac{\widehat{\mathfrak{Q}}_n(z)}{\mathfrak{L}_n(z)} - \frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)}\right| \rightrightarrows 0, \quad \forall z \in K.$$

Proof. For each fixed $n > m$, we have that

$$\frac{\widehat{\mathfrak{Q}}_n(z) - \mathfrak{P}_n(z)}{\mathfrak{L}_n(z)} = \sum_{k=0}^m\left(\frac{\lambda_n}{\lambda_{n-k}} - 1\right)\frac{b_{n,n-k}}{c_n^k}\,\frac{\mathfrak{L}_{n,n-k}(z)}{\mathfrak{L}_n(z)}. \tag{42}$$

As $\lambda_n = -n$ in the Laguerre case and $\lambda_n = -2n$ in the Hermite case, then for each k fixed, $k = 1,\ldots,m$,

$$\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n-k}} = 1. \tag{43}$$

From (30), (42), (43) and Proposition 5 we deduce the lemma. □

Proof. [Theorem 1]

a) From [18, (5.1.14),(5.5.10)] we have that $\mathfrak{L}'_{n,n-k} = (n-k)\widetilde{\mathfrak{L}}_{n,n-1-k}$, where

$$\widetilde{\mathfrak{L}}_{n,n-1-k} = \begin{cases} c_n^{-(n-1-k)} L_{n-1-k}^{\alpha+1}(c_n z), & \text{Laguerre case,} \\ c_n^{-(n-1-k)} H_{n-1-k}(c_n z), & \text{Hermite case.} \end{cases}$$

Let us define

$$\begin{aligned} d\widetilde{w}(x) &= \begin{cases} dw_L^{\alpha+1}(x), & \text{Laguerre case,} \\ dw_H(x), & \text{Hermite case.} \end{cases} \\ dw_n(x) &= \begin{cases} c_n^{-1} dw_L^{\alpha}(c_n x), & \text{Laguerre case,} \\ c_n^{-1} dw_H(c_n x), & \text{Hermite case.} \end{cases} \\ d\widetilde{w}_n(x) &= \begin{cases} c_n^{-1} dw_L^{\alpha+1}(c_n x), & \text{Laguerre case,} \\ c_n^{-1} dw_H(c_n x), & \text{Hermite case.} \end{cases} \end{aligned}$$

Notice that $\{\mathfrak{L}_{n,n-k}\}_{k=0}^{n}$ and $\{\widetilde{\mathfrak{L}}_{n,n-k}\}_{k=0}^{n}$ are monic orthogonal polynomials with respect to $w_n, \widetilde{w}_n$ respectively, hence, from [7, (11)], we have that the sequences $\{\mathfrak{L}_{n,n-k}\}_{n=0}^{\infty}$ and $\{\widetilde{\mathfrak{L}}_{n,n-k}\}_{n=0}^{\infty}$ for every $k = 0, \dots, m$ satisfy that

$$\lim_{n\to\infty} \|w_n \mathfrak{L}_{n,n-k}\|_{L^2(\Delta)}^{1/n} = e^{-F_w}, \lim_{n\to\infty} \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^2(\Delta)}^{1/n} = e^{-F_w}, \tag{44}$$

where F_w is the modified Robin constant for the weights $w, \widetilde{w}$ (or the extremal constant according to the terminology of [7]) and $\|.\|_{L^2(\Delta)}$ denotes the L^2–norm with the Lebesgue measure with support on Δ.

From [8, Ths. 1 & 2] we have that

$$\begin{aligned} \|w_n \mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)} &\leq K_1 n^{\beta} \|w_n \mathfrak{L}_{n,n-k}\|_{L^2(\Delta)}, \\ \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^\infty(\Delta)} &\leq K_2 n^{\beta} \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^2(\Delta)}, \end{aligned} \tag{45}$$

where K_1, K_2 are constants that do not depend on n, $\beta = 1/2$ for the Laguerre case, and $\beta = 1/4$ for the Hermite case. Using (44), (45), and [7, (11)] we obtain that

$$\lim_{n\to\infty} \|w_n \mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)}^{1/n} = e^{-F_w}, \lim_{n\to\infty} \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^\infty(\Delta)}^{1/n} = e^{-F_w}. \tag{46}$$

Then we have

$$\begin{aligned} \|w_n \widehat{\mathfrak{Q}}_n\|_{L^\infty(\Delta)} &\leq \sum_{k=0}^{m} \left| \frac{\lambda_n\, b_{n,n-k}}{c_n^k \lambda_{n-k}} \right| \|w_n \mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)} \\ &\leq \left| \frac{(m+1)\lambda_n\, b_{n,n-k^*(n)}}{c_n^{k^*(n)} \lambda_{n-k^*(n)}} \right| \|w_n \mathfrak{L}_{n,n-k^*(n)}\|_{L^\infty(\Delta)}, \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{w}_n \widehat{\mathfrak{Q}}'_n\|_{L^\infty(\Delta)} &\leq \sum_{k=0}^{m} \left| \frac{(n-k)\lambda_n\, b_{n,n-k}}{c_n^k \lambda_{n-k}} \right| \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-1-k}\|_{L^\infty(\Delta)} \\ &\leq \left| \frac{(m+1)(n-k^{**}(n))\lambda_n\, b_{n,n-k^{**}(n)}}{c_n^{k^{**}(n)} \lambda_{n-k^{**}(n)}} \right| \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-1-k^{**}(n)}\|_{L^\infty(\Delta)}, \end{aligned}$$

where $\|.\|_{L^\infty(\Delta)}$ denotes the sup norm and $k^*(n), k^{**}(n)$ denote positive integer numbers such that the following equalities hold

$$\left|\frac{\lambda_n\, b_{n,n-k^*(n)}}{c_n^{k^*(n)}\lambda_{n-k^*(n)}}\right| \|w_n\mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)} = \max_{k=0,\dots,m}\left|\frac{\lambda_n\, b_{n,n-k}}{c_n^k\lambda_{n-k}}\right| \ \|w_n\mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)},$$

$$\begin{aligned}&\left|\frac{(n-k^{**}(n))\lambda_n\, b_{n,n-k^{**}(n)}}{c_n^{k^{**}(n)}\lambda_{n-k^{**}(n)}}\right| \ \|\widetilde{w}_n\widetilde{\mathfrak{L}}_{n,n-1-k^{**}(n)}\|_{L^\infty(\Delta)}\\ &= \max_{k=0,\dots,m}\left|\frac{(n-k)\lambda_n\, b_{n,n-k}}{c_n^k\lambda_{n-k}}\right| \ \|\widetilde{w}_n\widetilde{\mathfrak{L}}_{n,n-k}\|_{L^\infty(\Delta)}.\end{aligned}$$

From these last inequalities and (46) we deduce that

$$\lim_{n\to\infty}\left(\|w_n\widehat{\mathfrak{Q}}_n\|_{L^\infty(\Delta)}\right)^{1/n} = e^{-F_w},\ \lim_{n\to\infty}\left(\|\widetilde{w}_n\widehat{\mathfrak{Q}}_n'\|_{L^\infty(\Delta)}\right)^{1/n} = e^{-F_w},$$

Therefore, if ν_n, δ_n denote the root counting measure of $\widehat{\mathfrak{Q}}_n$ and $\widehat{\mathfrak{Q}}_n'$ respectively, from [10, Th. 1.1] we deduce that $\nu_n \overset{*}{\longrightarrow} \nu_w$, $\delta_n \overset{*}{\longrightarrow} \nu_w$ in the weak star sense.

b) From Lemma 9, if ε is sufficiently small and $K\subset\mathbb{C}\setminus\Delta_c$ is a compact subset, for all n sufficiently large we have that, for some positive constant C,

$$C-\varepsilon \le \left|\frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)}\right| - \varepsilon \le \left|\frac{\widehat{\mathfrak{Q}}_n(z)}{\mathfrak{L}_n(z)}\right|.$$

From this fact and from Lemma 6 we deduce that the set of accumulation points is contained on Δ_c and from *a*) of the present theorem we deduce that the set of accumulation points of the zeros of $\widehat{\mathfrak{Q}}_n$ is Δ_c. □

Proof. [Theorem 2] From *b*) of Theorem 1 we deduce that for the Laguerre case

$$\lim_{n\to\infty}\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)} = \lim_{n\to\infty}\frac{\widehat{Q}_n''(c_nz)}{\widehat{Q}_n'(c_nz)} = \frac{1}{2\pi}\int_0^1 \frac{1}{z-t}\sqrt{\frac{1-t}{t}}dt = \frac{1}{2}\left(1-\sqrt{1-1/z}\right),$$

and for the Hermite case

$$\lim_{n\to\infty}\frac{\widehat{Q}_n'(c_nz)}{c_n\widehat{Q}_n(c_nz)} = \lim_{n\to\infty}\frac{\widehat{Q}_n''(c_nz)}{c_n\widehat{Q}_n'(c_nz)} = \frac{1}{\pi}\int_{-1}^1 \frac{\sqrt{1-t^2}}{z-t}\,dt = z\left(1-\sqrt{1-1/z^2}\right),$$

on compact subsets $K\subset\mathbb{C}\setminus\Delta_c$. From (6) and the preceding relations we have for the Laguerre case

$$\frac{P_n(c_nz)}{\widehat{Q}_n(c_nz)} \quad = \quad \frac{zc_n}{\lambda_n}\frac{\widehat{Q}_n''(c_nz)}{\widehat{Q}_n'(c_nz)}\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)} + \left(\frac{1+\alpha-c_nz}{\lambda_n}\right)\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)}, \tag{47}$$

and for the Hermite case

$$\frac{P_n(c_nz)}{\widehat{Q}_n(c_nz)} \quad = \quad \frac{1}{2}\frac{1}{\lambda_n}\frac{\widehat{Q}_n''(c_nz)}{\widehat{Q}_n'(c_nz)}\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)} - \left(\frac{c_nz}{\lambda_n}\right)\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)}. \tag{48}$$

Taking limits in (47) and (48) we obtain (16). Relation (17) follows from (14) and (16). □

4. The polynomial $\mathfrak{Q}_n$

Some basic properties of the zeros of $\mathfrak{Q}_n$ follow directly from (1) and (2). For example, the multiplicity of the zeros of $\mathfrak{Q}_n$ is at most 3, a zero of multiplicity 3 is also a zero of $\mathfrak{P}_n$ and a zero of multiplicity 2 is a critical point of $\widehat{\mathfrak{Q}}_n$. In the next lemma we prove conditions for the boundedness of the zeros of $\mathfrak{Q}_n$ and determine their asymptotic behavior.

Lemma 11. *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m \in \mathbb{N}$ and define for $z \in \mathbb{C}$, $\mathfrak{D}(z) = \sup_{x\in\Delta_c} |z-x|$. If $\{\zeta_n\}_{n=m+1}^{\infty}$ is a sequence of complex numbers with limit $\zeta \in \mathbb{C}$, then for every $d > 1$ there is a positive number N_d, such that $\{z \in \mathbb{C} : \mathfrak{Q}_n(z) = 0\} \subset \{z \in \mathbb{C} : |z| \leq \mathfrak{D}(\zeta) + d\}$ whenever $n > N_d$.*

Proof. As $\mathfrak{Q}_n(z) = 0$ then $\widehat{\mathfrak{Q}}_n(z) = \widehat{\mathfrak{Q}}_n(\zeta_n)$. From Gauss–Lucas theorem (cf. [16, §2.1.3]), it is known that the critical points of $\widehat{\mathfrak{Q}}_n$ are in the convex hull of its zeros and from *b*) of Theorem 1 the zeros of the polynomials $\{\widehat{\mathfrak{Q}}_n\}_{n=m+1}^{\infty}$ accumulate on Δ_c. Hence, from the *bisector theorem* (see [16, §5.5.7]) $|z| \leq \mathfrak{D}(\zeta_n) + 1$ and the lemma is established. □

We are now ready to prove Theorem 3.

Proof. [Theorem 3] From Lemma 11 we have that the zeros of $\mathfrak{Q}_n$ are located in a compact set. From (15) the zeros of $\mathfrak{Q}_n$ satisfy the equation

$$\left|\widehat{\mathfrak{Q}}_n(z)\right|^{\frac{1}{n}} = \left|\widehat{\mathfrak{Q}}_n(\zeta_n)\right|^{\frac{1}{n}} . \tag{49}$$

If $z \in \mathbb{C} \setminus \Delta_c$, taking limit when $n \to \infty$, from Lemma 11, and using (17) of Theorem 2 on both sides of (49), we have that the zeros of the sequence of polynomials $\{\mathfrak{Q}_n\}_{n=m+1}^{\infty}$ cannot accumulate outside the set

$$\{z \in \mathbb{C} : \Psi(z) = \Psi(\zeta)\} \bigcup \Delta_c.$$

To verify the second statement of the theorem, note that if z is a zero of $\mathfrak{Q}_n$, from (15) we get

$$\prod_{k=1}^{n} \left|\frac{z-\widehat{x}_{n,k}}{\zeta_n - \widehat{x}_{n,k}}\right| = 1, \text{ where } \widehat{x}_{n,k} \text{ are the zeros of } \widehat{\mathfrak{Q}}_n. \tag{50}$$

Let $\mathcal{V}_\varepsilon(\Delta_c)$ be the ε-neighborhood of Δ_c defined as $\mathcal{V}_\varepsilon(\Delta_c) = \{z \in \mathbb{C} : \mathfrak{d}(z) < \epsilon\}$, as $\lim_{n\to\infty} \zeta_n = \zeta$, then for all $\varepsilon > 0$ there is a $N_\varepsilon > 0$ such that $|\mathfrak{d}(\zeta_n) - \mathfrak{d}(\zeta)| < \varepsilon$ whenever $n > N_\varepsilon$.

If $\mathfrak{d}(\zeta) > 2$, let us choose $\varepsilon = \varepsilon_\zeta = \dfrac{1}{2}\left(\mathfrak{d}(\zeta) - 2\right)$ and suppose that there is a $z_0 \in \mathcal{V}_{\varepsilon_\zeta}(\Delta_c)$ such that $\mathfrak{Q}_n(z_0) = 0$ for some $n > N_{\varepsilon_\zeta}$. Hence

$$\prod_{k=1}^{n} \left|\frac{z_0-\widehat{x}_{n,k}}{\zeta_n - \widehat{x}_{n,k}}\right| < \left(\frac{2+\varepsilon_\zeta}{\mathfrak{d}(\zeta_n)}\right)^n < 1, \tag{51}$$

which is a contradiction with (50), hence $\{z \in \mathbb{C} : \mathfrak{Q}_n(z) = 0\} \bigcap \mathcal{V}_{\varepsilon_n}(\Delta_c) = \varnothing$ for all $n > N_{\varepsilon_\zeta}$, i.e. the zeros of $\mathfrak{Q}_n$ cannot accumulate on $\mathcal{V}_{\varepsilon_\zeta}(\Delta_c)$.

From (15) it is straightforward that a multiple zero of $\mathfrak{Q}_n$ is also a critical point of $\widehat{\mathfrak{Q}}_n$. But, from b) of Theorem 1 and the Gauss–Lucas theorem the set of accumulation points of $\widehat{\mathfrak{Q}}_n$ is Δ_c, where we have that for n sufficiently large the zeros of $\mathfrak{Q}_n$ are simple. □

Proof. [Theorem 4]
1.- Let us prove first that

$$\frac{\mathfrak{Q}_n(z)}{\widehat{\mathfrak{Q}}_n(z)} = 1 - \frac{\widehat{\mathfrak{Q}}_n(\zeta_n)}{\widehat{\mathfrak{Q}}_n(z)} \underset{n\to\infty}{\rightrightarrows} 1, \tag{52}$$

uniformly on compact subsets K of the set $\{z \in \mathbb{C} : |\Psi(z)| > |\Psi(\zeta)|\}$. In order to prove (52) it is sufficient to show that

$$\frac{\widehat{\mathfrak{Q}}_n(\zeta_n)}{\widehat{\mathfrak{Q}}_n(z)} \underset{n\to\infty}{\rightrightarrows} 0, \tag{53}$$

uniformly on K.

From [6] and Lemmas 8, 9, we have that for all n large enough it is possible to find constants c^*, c such that

$$c^* \leq \left|\frac{\mathfrak{P}_n(z)}{\Psi^n(z)}\right| \leq c, \tag{54}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_c$.Then we have

$$\left|\frac{\widehat{\mathfrak{Q}}_n(\zeta_n)}{\widehat{\mathfrak{Q}}_n(z)}\right| = \left|\frac{\widehat{\mathfrak{Q}}_n(\zeta_n)}{\mathfrak{P}_n(\zeta_n)}\right| \left|\frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}_n(z)}\right| \left|\frac{\mathfrak{P}_n(\zeta_n)}{\Psi^n(\zeta_n)}\right| \left|\frac{\Psi^n(z)}{\mathfrak{P}_n(z)}\right| \left|\left(\frac{\Psi(\zeta_n)}{\Psi(z)}\right)\right|^n.$$

From (16) of Theorem 2 and (54) the first four factors on the right hand side of the previous formula are bounded; meanwhile, the last factor tends to 0 when $n \to \infty$, and we get (53). Finally, the assertion 1 is straightforward from (16) of Theorem 2.

2.- For the assertion 2 of the theorem it is sufficient to prove that

$$\frac{\mathfrak{Q}_n(z)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} = \frac{\widehat{\mathfrak{Q}}_n(z)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} - 1 \underset{n\to\infty}{\rightrightarrows} -1, \tag{55}$$

uniformly on compact subsets K of the set $\{z \in \mathbb{C} : |\Psi(z)| < |\Psi(\zeta)|\} \setminus \Delta_c$. Note that

$$\frac{\widehat{\mathfrak{Q}}_n(z)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} = \frac{\widehat{\mathfrak{Q}}_n(z)}{\mathfrak{P}_n(z)} \frac{\mathfrak{P}_n(\zeta_n)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} \frac{\mathfrak{P}_n(z)}{\Psi^n(z)} \frac{\Psi^n(\zeta_n)}{\mathfrak{P}_n(\zeta_n)} \left(\frac{\Psi(z)}{\Psi(\zeta_n)}\right)^n.$$

Now, the first part of the assertion 2 is straightforward from (16) of Theorem 2 and (54).

If $\mathfrak{d}(\zeta) > 2$, let $\mathcal{V}_\varepsilon(\Delta_c) = \{z \in \mathbb{C} : \mathfrak{d}(z) < \epsilon\}$ be a ε–neighborhood of Δ_c, where $\varepsilon = \varepsilon_\zeta = \dfrac{\mathfrak{d}(\zeta)}{2} - 1$. By the same reasoning used to deduce (51) we get that

$$\left|\frac{\widehat{\mathfrak{Q}}_n(z)}{\widehat{\mathfrak{Q}}_n(\zeta_n)}\right| < \kappa^n, \quad \text{for all } z \in \mathcal{V}_\varepsilon(\Delta_c), \kappa < 1. \tag{56}$$

Hence from the first part of the assertion 2 and (56) we get the second part of the assertion 2. □

5. A fluid dynamics model

In this section we show a hydrodynamical model for the zeros of the orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$. In [4], we gave a hydrodynamic interpretation for the critical points of orthogonal polynomials with respect to a Jacobi differential operator.

Let us consider a flow of an incompressible fluid in the complex plane, due to a system of n *different* points ($n > 1$) fixed at w_i, $1 \leq i \leq n$. At each point w_i of the system there is defined a complex potential $\mathcal{V}_i$, which for the Laguerre case equals to the sum of a *source(sink)* with a fixed strength $\Re[c_i]$ plus a *vortex* with a fixed strength $\Im[c_i]$ plus a *uniform stream* U_i at infinity. Here c_i and d_i are fixed complex numbers which depend on the position of the remaining points $\{w_i\}_{i=1}^n$, see [9, Ch. VIII] for the terminology. We shall call n *system* to the set of the n points fixed at w_i with its respective potential of velocities.

Define the functions

$$f_i(w_1, \dots, w_n) = \frac{R_n''(w_i)}{R_n'(w_i)}, \quad i = 1, \dots, n \quad \text{where} \quad R_n(z) = \prod_{i=1}^n (z - w_i).$$

The complex potentials $\mathcal{V}_L$ (Laguerre case) or $\mathcal{V}_H$ (Hermite case) at any point z (see [5, Ch. 10]), by the principle of superposition of solutions, are given by

$$\mathcal{V}_L(z)=\sum_{i=1}^{n}\mathcal{V}_{L,i}=\sum_{i=1}^{n}\left(-z+(1+\alpha-w_i)\log\left(z-w_i\right)+\left(z+w_i\log\left(z-w_i\right)\right)f_i(w_1,\ldots,w_n)\right), \tag{57}$$

and

$$\mathcal{V}_H(z)=\sum_{i=1}^{n}\mathcal{V}_{H,i}=\sum_{i=1}^{n}\left(-z+\frac{1}{2}(f_i(w_1,\ldots,w_n)-2w_i)\log\left(z-w_i\right)\right). \tag{58}$$

From a complex potential $\mathcal{V}$, a complex velocity $\mathbf{V}$ can be derived by differentiation ($\mathbf{V}(z)=\frac{d\mathcal{V}}{dz}$). A standard problem associated with the complex velocity is to find the zeros, that correspond to the set of *stagnation points*, i.e. points where the fluid has zero velocity.

We are interested in the problem: Build an n system (location of the points $w_1,\ldots,w_n$) such that the stagnation points are at preassigned points with *nice* properties. As it is well known, the zeros of the orthogonal polynomials with respect to a finite positive Borel measures on $\mathbb{R}$ have a rich set of *nice* properties (cf. in [18, Ch. VI]), and will be taken as preassigned stagnation points. Here we consider $\mu\in\mathcal{P}_1[\mathbb{R}_+]$ or $\mu\in\widetilde{\mathcal{P}}_2[\mathbb{R}]$. In the next paragraph we establish the statement of the problem for both Laguerre and Hermite cases.

Problem. *Let $\{x_1,\ldots,x_n\}$ be the set of zeros of the nth orthogonal polynomial P_n ($n>1$ for the Laguerre case and $n>2$ for the Hermite) with respect to a measure $\mu\in\mathcal{P}_1[\mathbb{R}_+]$ or $\mu\in\widetilde{\mathcal{P}}_2[\mathbb{R}]$. Suppose a flow is given, with complex potential $\mathcal{V}_L$ (Laguerre case) or $\mathcal{V}_H$ (Hermite). Build an n system (location of the points $w_1,\ldots,w_n$) such that the stagnation points are attained at the points $z=x_i$, with $i=1,2,\ldots,n$.*

Consider first the Laguerre case. If x_k ($k=1,\ldots,n.$) are stagnation points then

$$\frac{\partial\mathcal{V}_L}{\partial z}(x_k)=(1+\alpha-x_k)\sum_{i=1}^{n}\frac{1}{x_k-w_i}+x_k\sum_{i=1}^{n}\frac{R_n''(w_i)}{R_n'(w_i)(x_k-w_i)}=0. \tag{59}$$

We are looking for a solution $R_n(z)=\prod_{i=1}^{n}(z-w_i)$, with $w_i\neq w_j\neq x_k,\forall i,j,k,\ i\neq j$, such that (59) holds. This assumption implies that the sum in the second term of the left hand side of expression (59) is the partial-fraction decomposition of $\dfrac{R_n''}{R_n}$ evaluated at $x=x_k$. Therefore, (59) is equivalent to

$$x_kR_n''(x_k)+(1+\alpha-x_k)R_n'(x_k)=0,\quad k=1,\,2,\,\ldots,n.$$

Note that $xR_n''(x)+(1+\alpha-x)R_n'(x)$ is a polynomial of degree n, with leading coefficient λ_n that vanishes at the zeros of P_n, i.e.

$$xR_n''(x)+(1+\alpha-x)R_n'(x)=\lambda_nP_n(x). \tag{60}$$

Observe that expression (60) is equivalent to (6). From Proposition 4, the zeros of $\widehat{Q}_n,\widehat{Q}_n'$ are real, simple and $Q_n'(x_k)\neq 0$. Therefore, $R_n=\widehat{Q}_n$ is a solution. Hence, an answer to our problem yields the n points as the n zeros of the polynomial $\widehat{Q}_n$.

For the Hermite case we have a similar situation. Thus, if x_k is a stagnation point, $\dfrac{\partial\mathcal{V}_H}{\partial z}(x_k)=0$, which gives

$$x_k\sum_{i=1}^{n}\frac{1}{x_k-w_i}-\frac{1}{2}\sum_{i=1}^{n}\frac{R_n''(w_i)}{R_n'(w_i)(x_k-w_i)}=0,\quad k=1,\,2,\,\ldots,n. \tag{61}$$

Again, we can deduce that the expression (61) equals to $\dfrac{1}{2}R_n''(x_k)-x_kR_n'(x_k)=0$, for $k=1,\ldots,n$.

Note that $\frac{1}{2}R''_n(x) - xR'_n(x)$ is a polynomial of degree n, with leading coefficient λ_n that vanishes at the zeros of P_n, i.e.

$$\frac{1}{2}R''_n(x) - xR'_n(x) = \lambda_n P_n(x). \tag{62}$$

Therefore, the expression (62) is equivalent to (6). From Proposition 4, the zeros of $\widehat{Q}_n, \widehat{Q}'_n$ are real and simple and $Q'_n(x_k) \neq 0$, which implies that $R_n = \widehat{Q}_n$ is a solution to our problem. As a conclusion,

> **Answer.** *The flow of an incompressible two–dimensional fluid, due to n points with complex potential $\mathcal{V}_L$ or $\mathcal{V}_H$, located at the zeros of the nth orthogonal polynomial $\widehat{Q}_n$ with respect to $(\mathcal{L}, \mu)$, with $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$ has its n stagnation points at the n zeros of the nth orthogonal polynomial $\widehat{Q}_n$.*

It would be interesting to consider the uniqueness of the solution obtained. In other words, what could be said about the solutions of the form $Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n)$ and to extend this model to more general classes of measures μ. It would be also of interest to decide if these stagnation or equilibrium points are stable.

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